CS 133 : Automata Theory and Computability

Lecture Slides

Turing Machines and Undecidability

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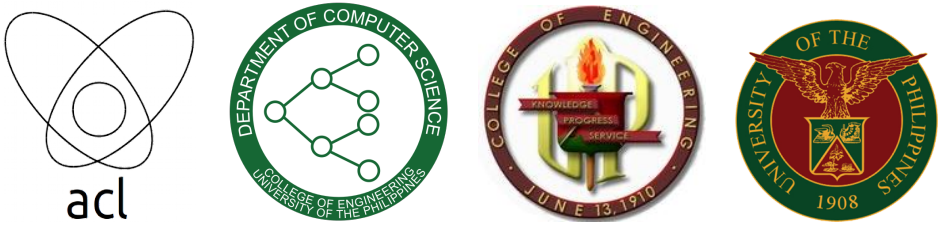

Algorithms and Complexity Laboratory
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University of the Philippines Diliman

Day 23

Recalling undecidability

Reduction

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Either case allows a contradiction, ergo, assumption $L_d = L(M_j)$ is false, i.e. M_j cannot exist.

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A popular variant of L_u is the **Halting problem**. Also, since L_u is not decidable then so is software verification *in general*!

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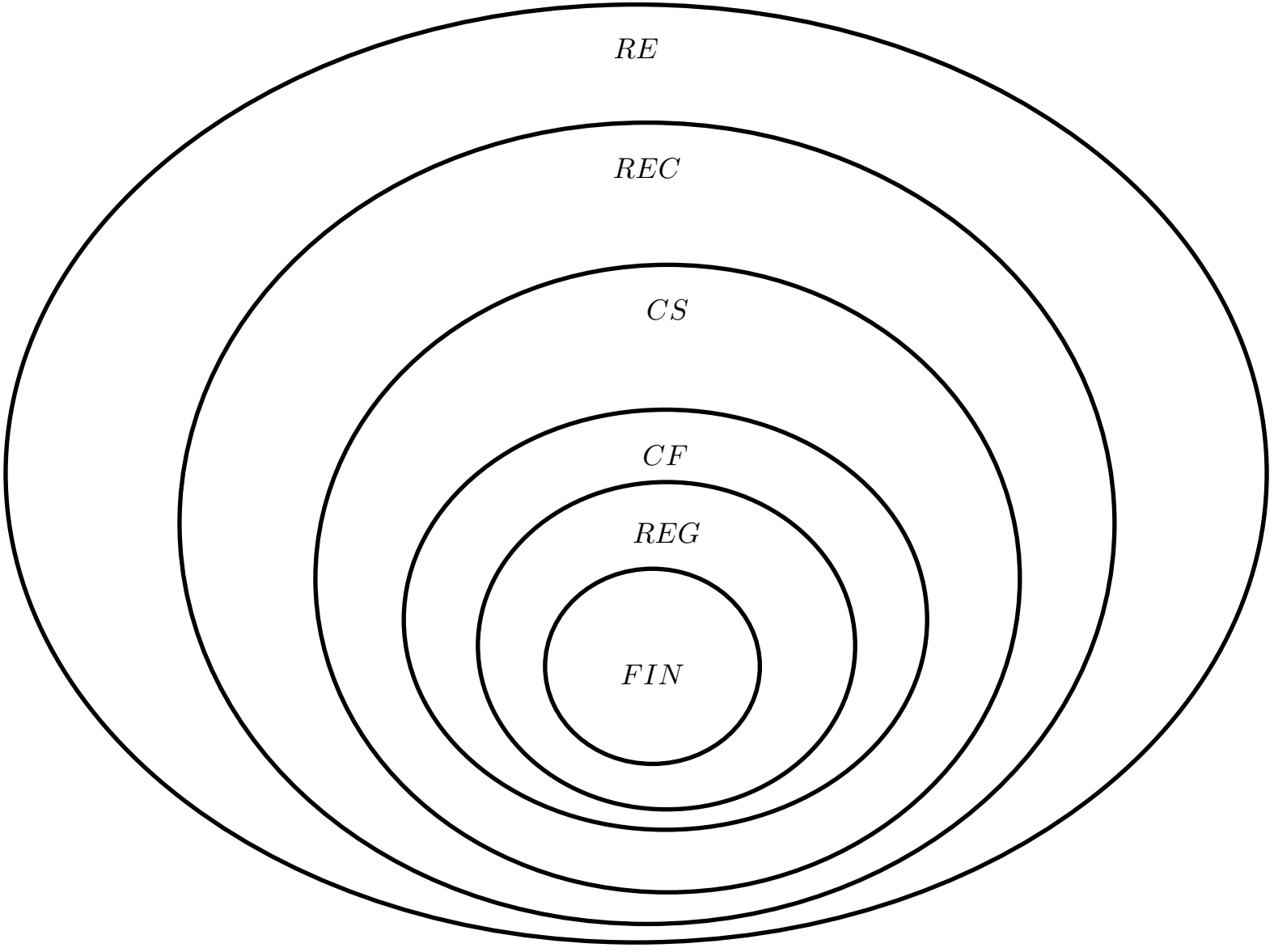
Chomsky hierarchy: complete!



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What about "more general" problems, e.g. transducers or computing functions?
Convert the following problems to their language counterparts: Given $a, b \in \mathbb{N}$, output $c = a + b$; Given $\Sigma = \{x, y, z\}$ and $w \in \Sigma^*$, output $w' = x^{|w|_x} y^{|w|_y} z^{|w|_z}$.

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If $L(M) = \emptyset$, then no x leads to acceptance.

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How to prove $L_e \notin RE$? **Homework!**

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Example

$$\left\{ \left[\frac{a}{ab}\right], \left[\frac{b}{ca}\right], \left[\frac{ca}{a}\right], \left[\frac{abc}{c}\right], \left[\frac{ab}{b}\right] \right\}.$$

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Before getting into the construction, for convenience, we make the modifications (resulting to the modified PCP or MPCP):

- we assume that M on w never attempts to move its head off the left-hand end of the tape
- if $w = \varepsilon$, we use the string $\sqcup$ in place of w
- we modify PCP to require that a match starts with the first domino, $\left[\frac{t_1}{b_1}\right]$

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Part 1. Put $\left[\frac{\#}{\#q_1 w_1 w_2 \cdots w_n \#}\right]$

Part 2. For every $a, b \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{reject}$, if $\delta(q, a) = (r, b, R)$, put $\left[\frac{qa}{br}\right]$ into P'.

Part 3. For every $a, b, c \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{reject}$, if $\delta(q, a) = (r, b, L)$, put $\left[\frac{cqa}{rcb}\right]$ into P'.

Part 4. For every $a \in \Gamma$, put $\left[\frac{a}{a}\right]$ into P'.

Part 5. Put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P'.

Part 6. For every $a \in \Gamma$, put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P'.

Part 7. Finally, we add the domino $\left[\frac{q_{accept}\#\#}{\#}\right]$ into P'.

Fin (wakas)

Thanks for the attention.

Questions?

Reading assignment(s)

[Sipser 2005] Chapter 5.1, 5.2, or

[Hopcroft et al 2001] Chapter 8.1.3, 9.1, 9.2, 9.3, 9.4

References:

[Sipser 2005] M. Sipser. *Introduction to the Theory of Computation*: 2ed. PWS Publishing Company, 2005.

[Hopcroft, Ullman 1979] J. Hopcroft, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 1979.

[Hopcroft et al 2001] J. Hopcroft, R. Motwani, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 2001.