CS 133 : Automata Theory and Computability

Lecture Slides

Turing Machines and Decidability

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Day 21

Decidability

Recursive Problems Concerning Regular Languages

Recursive Problems Concerning CFLs

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- We divide problems that can be solved by a TM into 2 classes:
  - **Class** *REC*: recursive or decidable languages, i.e. those that have an **algorithm** (i.e. a **TM** that **halts**, either accepting or rejecting input)
  - **Class** *RE*: recursively enumerable or recognizable languages, i.e. those that have an **algorithm** (i.e. TM) that **halt** if input is **accepted**, but **may loop forever** if input is **rejected**.

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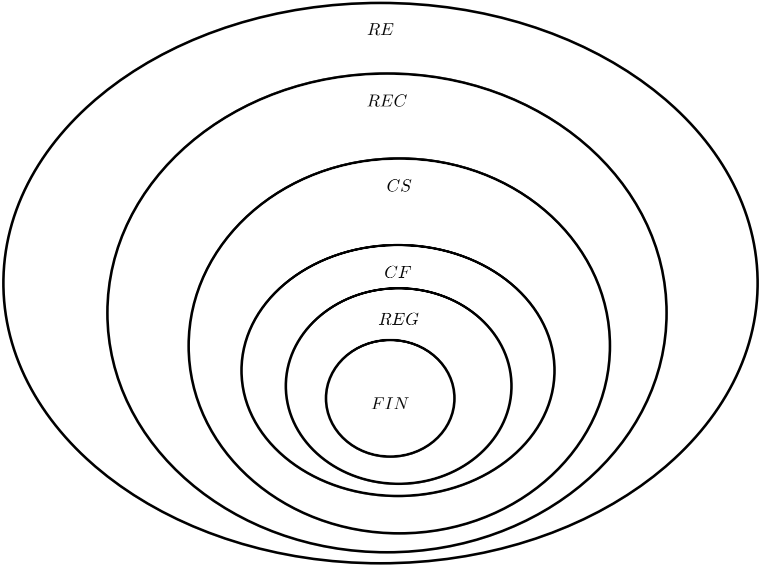
- Relationship of REC and RE to closure properties?

Chomsky hierarchy so far...



Decidability

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$M =$ "On input $\langle B, w\rangle$, where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, accept. If it ends in a nonaccepting state, reject."

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$N =$ "On input $\langle B, w\rangle$, where B is an NFA and w is a string:

1. Convert NFA B to an equivalent DFA C.
2. Run TM M on input $\langle C, w\rangle$.
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Running TM M in stage 2 means incorporating M into the design of N as a subprocedure.

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$\triangleright$ shows DFA, NFA or regular expression are equivalent because the machine is able to convert one form of encoding to another.

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$T =$ "On input $\langle A \rangle$, where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:
   ▷ Mark any state that has a transition coming into it from any state that is already marked.
3. If no accept state is marked, accept; otherwise, reject."

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▷ Testing for emptiness.

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We construct a new DFA C from A and B, where C accepts only those strings that are accepted by either A or B but not by both.

$F =$ "On input $\langle A, B\rangle$, where A and B are DFAs :

1. Construct DFA C as described.
2. Run TM T on input $\langle C\rangle$
3. If T accepts, accept; otherwise, reject."

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$R =$ "On input $\langle G \rangle$, where G is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variables get marked:
   $\triangleright$ Mark any variable A where G has a rule $A \to U_1 U_2 \cdots U_k$ and each symbol $U_1, \cdots, U_k$ has already been marked.
3. If the start variable is not marked, accept; otherwise, reject."

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Exercise: If G is in Chomsky normal form, any derivation of w has $2n - 1$ steps, where n is the length of w.

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$S =$ "On input $\langle G, w\rangle$, where G is a CFG and w is a string:

1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n - 1$ steps, where n is the length of w, except if $n = 0$, then instead list all derivations with 1 step.
3. If any of these derivations generate w, accept; if not, reject."

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Let G be a CFG for A and design a TM M_G that decides A.

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▷ Testing or checking for **membership** in CFL is recursive (decidable).

Fin (wakas)

Thanks for the attention.

Questions?

Work on examples and exercises in:

[Sipser 2005] Chapter 4.1 , or

[Hopcroft et al 2001] Chapter 4.2, 7.3

References:

[Sipser 2005] M. Sipser. *Introduction to the Theory of Computation*: 2ed. PWS Publishing Company, 2005.

[Hopcroft, Ullman 1979] J. Hopcroft, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 1979.

[Hopcroft et al 2001] J. Hopcroft, R. Motwani, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 2001.

[Hernandez 2014] CS133 lecture slides of N.H.S. Hernandez, 2014