

CS133 : Automata Theory and Computability

LECTURE SLIDES

Time Complexity

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Day 26

Class NP

or problems solvable in polynomial time with a 1-tape NTM. More formally

Definition

Language L is in class **NP** if there is a NTM M such that $L = L(M)$ and M has polynomial run time $t(n)$.

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More real-world applications: Cryptography!

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Method to prove **NP**-completeness? Polynomial time reduction! (\leq_p)

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Theorem: $3SAT \leq_p CLIQUE.$

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*Once we have one **NP**-complete problem, we may obtain others by poly time reduction from it!*

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▷ Show that every language A in **NP** is poly time reducible to *SAT*.

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Pick any language A in **NP**.

Let N be an NTM that decides A in $O(n^k)$ time for some constant k .

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n^k rows	#	q_0	w_1	w_2	...	w_n	\sqcup	...	\sqcup	#	Initial config	
	#									#	2nd config	
	#									#		
												a 2×3 window
	#										#	n^k th config
	n^k columns											

Problem of determining if N accepts $w \iff$ problem of determining if an accepting tableau for N on w exists.

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- For each $1 \leq i, j \leq n^k$ and $s \in C$, add variable $x_{i,j,s}$.
 $x_{i,j,s} = 1$ if $cell[i, j]$ of tableau contains s .

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 $x_{i,j,s} = 1$ if $cell[i, j]$ of tableau contains s .
- The formula $\phi_{N,w}$ is the AND of four parts, i.e.

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$$\phi_{cell} = \bigwedge_{1 \leq i, j \leq n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{s, t \in C, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right) \right]$$

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- 1st clause: at least one variable $x_{i,j,s} = 1$ corresponds to $\text{cell}[i, j] = s$.

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- Both clauses: we ensure any satisfying assignment of $\phi_{N,w}$ corresponds to one symbol in each cell of tableau.

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Formula ϕ_{start} ensures that 1st row of tableau is the *initial configuration* of N on $w = w_1 w_2 \dots w_n$:

$$\begin{aligned} \phi_{\text{start}} = & x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \dots \wedge x_{1,n+2,w_n} \wedge x_{1,n+3,\sqcup} \\ & \wedge \dots \wedge x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#} \end{aligned}$$

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Formula ϕ_{accept} ensure that an *accepting configuration* occurs in tableau.

$$\phi_{\text{accept}} = \bigvee_{1 \leq i, j \leq n^k} x_{i,j,q_{\text{accept}}}$$

$$\phi_{N,w} = \phi_{cell} \wedge \phi_{start} \wedge \phi_{move} \wedge \phi_{accept}.$$

Finally, formula ϕ_{move} ensures row i of tableau *legally follows* from row $i - 1$ based on transitions of N . Use concept of *legal windows*.

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Replace the text 'the (i, j) window is legal' with the following formula:

$$\bigwedge_{\substack{a_1, a_2, \dots, a_6 \\ \text{is a legal window}}} (x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6})$$

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Since any A in **NP** reduces to SAT , SAT is **NP**-complete.

Back to 3SAT

Corollary

3SAT is NP-complete

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- ϕ_{move} is a big AND of subformulas, each of which is an OR of ANDs. Note: *Distributive laws state that we can replace an OR of ANDs with an equivalent AND of ORs.*
- If clause contains l variables, $(a_1 \vee a_2 \vee \cdots \vee a_l)$, we can replace it with the $l - 2$ clauses
 $(a_1 \vee a_2 \vee z_1) \wedge (\bar{z}_1 \vee a_3 \vee z_2) \wedge (\bar{z}_2 \vee a_4 \vee z_3) \wedge \cdots \wedge (\bar{z}_{l-3} \vee a_{l-1} \vee a_l)$.
- Now we have a formula in CNF with 3 variables per clause!

NP-completeness is widespread!

NP-complete problems are found in practically any field requiring computation, e.g.

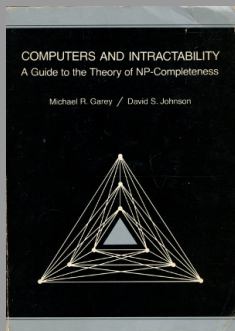
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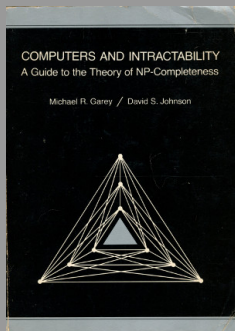


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If you cannot find a poly time algo for a problem, good idea to check if problem is **NP**-complete.

Corollary

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A vertex cover of an undirected graph $G = (V, E)$ is a set $V' \subseteq V$ where all nodes in V' “cover” all edges of G ,

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The vertex cover problem asks whether a graph contains a vertex cover of a size k :

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Need to show: $3SAT \leq_p VERTEX-COVER$.

Reduction maps a 3CNF ϕ to a graph G and number k , so ϕ is satisfiable only if G has a k -node vertex cover.

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Poly time reduction technique: from 3SAT to some language A , look for structures in A that can simulate variables and clauses in Boolean formulas.

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E.g. $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$.

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- Vertex cover must contain one node in each variable gadget and two in every clause gadget in order to cover all edges (edges of the variable and the edges within the clause gadgets).

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- Hence, ϕ has a satisfying assignment.

Theorem

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The reduction maps a 3CNF ϕ with variables x_1, \dots, x_l and clauses c_1, \dots, c_k into an instance of the *SUBSETSUM* problem $\langle S, t \rangle$ wherein the elements of S and the number t are the rows in the table on the next slide.

	1	2	3	4	...	l	c_1	c_2	...	c_k
y_1	1	0	0	0	...	0	1	0	...	0
z_1	1	0	0	0	...	0	0	0	...	0
y_2		1	0	0	...	0	0	1	...	0
z_2		1	0	0	...	0	1	0	...	0
\vdots					\ddots	\vdots	\vdots		\vdots	\vdots
y_l						1	0	0	...	0
z_l						1	0	0	...	0
g_1							1	0	...	0
h_1							1	0	...	0
g_2								1	...	0
h_2								1	...	0
\vdots									\ddots	\vdots
g_k										1
h_k										1
t	1	1	1	1	...	1	3	3	...	3

The reduction maps a 3CNF ϕ with variables x_1, \dots, x_l and clauses c_1, \dots, c_k into an instance of the *SUBSETSUM* problem $\langle S, t \rangle$ wherein the elements of S and the number t are the rows in the table on the previous slide, expressed in ordinary decimal notation.

The reduction maps a 3CNF ϕ with variables x_1, \dots, x_l and clauses c_1, \dots, c_k into an instance of the *SUBSETSUM* problem $\langle S, t \rangle$ wherein the elements of S and the number t are the rows in the table on the previous slide, expressed in ordinary decimal notation. Constructed table has $2l + 2k$ decimal numbers (rows) and the final row for t . Each decimal number has at most $l + k$ digits (columns).

- S contains one pair of numbers, y_i, z_i for each variable x_i in ϕ .
- The decimal representation is in two parts.
The left-hand part comprises a 1 followed by $l - i$ 0s.
The right-hand part contains one digit for each clause.
 - the j^{th} digit of y_i is 1 if clause c_j contains literal x_i
 - the j^{th} digit of z_i is 1 if clause c_j contains literal \bar{x}_i
- S also contains one pair of numbers g_j, h_j for each clause c_j .
These numbers are equal and consist of a 1 followed by $k - j$ 0s.
- The target number t , consists of l number of 1s followed by k number of 3s.

Suppose ϕ is satisfiable.

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We construct a subset of S as follows:

- Select y_i if $x_i = 1$. Select z_i otherwise.
 - so far, the sum contains a 1 in each of the first l digits and a number between 1 and 3 in each of the last k digits
- Select enough of the g and h numbers to bring each of the last k digits up to 3.
- Sum would then be equal to t .

	1	2	3	4	...	l	c_1	c_2	...	c_k
y_1	1	0	0	0	...	0	1	0	...	0
z_1	1	0	0	0	...	0	0	0	...	0
y_2		1	0	0	...	0	0	1	...	0
z_2		1	0	0	...	0	1	0	...	0
\vdots					\ddots	\vdots	\vdots		\vdots	\vdots
y_l						1	0	0	...	0
z_l						1	0	0	...	0
g_1							1	0	...	0
h_1							1	0	...	0
g_2								1	...	0
h_2								1	...	0
\vdots									\ddots	\vdots
g_k										1
h_k										1
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Suppose that a subset of S sums to t .

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Note: a “carry” into next column in the table never occurs when a subset of S is added, since each column sums up to at most 5.

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Note: a “carry” into next column in the table never occurs when a subset of S is added, since each column sums up to at most 5.

We make the satisfying assignment as follows:

- To get a 1 in each of the first l columns, the subset must have either y_i or z_i for each i but not both.
 - if the subset contains y_i , assign $x_i = 1$; otherwise, $x_i = 0$.
- For each of the final k columns, at most 2 can come from g and h . To sum up to 3, thus matching the digit (column) of t , at least 1 (for each of these columns) must come from some y_i or z_i in the subset.
 - if it is y_i , then x_i appears in c_j and $x_i = 1$, so c_j is satisfied.
 - if it is z_i , then \bar{x}_i appears in c_j and $x_i = 0$, so c_j is satisfied.
- Therefore, ϕ is satisfied.

More NP-complete problems

HAMPATH, *GRAPHCOLOR*, *KNAPSACK*, *LCSP*, Super Mario Bros., Candy Crush Saga, The Legend of Zelda, Pokémon et al.

https://en.wikipedia.org/wiki/List_of_NP-complete_problems

Fin (wakas)

Thanks for the attention.

Questions?

Reading assignment(s)

[Sipser 2005] Chapter 7.4, 7.5 or

[Hopcroft et al 2001] Chapter 10.2, 10.3,
10.4

References:

[Sipser 2005] M. Sipser. *Introduction to the Theory of Computation*: 2ed. PWS Publishing Company, 2005.

[Hopcroft, Ullman 1979] J. Hopcroft, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 1979.

[Hopcroft et al 2001] J. Hopcroft, R. Motwani, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 2001.



A pleasure and a privilege to have shared CS133 with you this semester.