CS 133 : Automata Theory and Computability

Lecture Slides

Time Complexity

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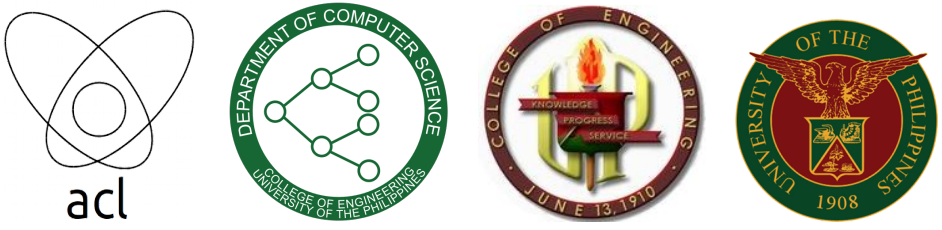

Algorithms and Complexity Laboratory
Department of Computer Science
College of Engineering
University of the Philippines Diliman

Day 25

Classes **P** and **NP**

NP-completeness

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Class P

or problems solvable in polynomial time with a 1-tape DTM. More formally

Definition

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$PATH \in \mathbf{P}$. (Use BFS)

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Applications: planning, logistics, microchip manufacturing, DNA sequencing, etc.

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Problems in **NP** have "slow" or "inefficient" algo.

Deciding $HAMPATH$

The following NTM decides $HAMPATH$ in nondeterministic polynomial time.

N_1 = "On input $\langle G, s, t \rangle$ where G is a digraph with nodes s and t:

1. Nondeterministically choose a list of m numbers $p_1, \ldots, p_m$, where m is the number of nodes in G and $1 \leq p_i \leq m$.
2. Check for repetitions in the list. If any are found, reject.
3. Check whether $s = p_1$ and $t = p_m$. If either fail, reject.
4. $\forall i, 1 \leq i \leq m-1$, check if (p_i, p_{i+1}) is an edge of G. If any are not, reject. Otherwise, all tests have passed, so accept."

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What about $\overline{HAMPATH}$?

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We use nondeterminism to *guess* a witness or a possibly correct solution to the problem in **NP**.

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Despite this, exact algos and heuristic algos can still take inordinate amount of run times in general.

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If $\mathbf{P} \neq \mathbf{NP}$, this means proving *no fast algorithm exists* to replace brute-force search .

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- practical: May prevent wasting time searching for a nonexistent poly time algo to solve a general problem.
  Note: book author Michael Sipser believes that $\mathbf{P} \neq \mathbf{NP}$*, so proving a problem is* ***NP****-complete is evidence of its nonpolynomiality.*

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SAT was the first problem to be identified to be **NP**-complete.
Method to prove **NP**-completeness?

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The function f is called the **polynomial time reduction** of A to B.

Polynomial Time Reduction

In computability theory we used reduction to show undecidability (without considering efficiency of reduction).
Considering efficiency of reduction: If problem A is **efficiently reducible** to problem B, an efficient solution to B can be used to solve A efficiently.

Definition

A function $f : \Sigma^* \to \Sigma^*$ is a **poly time computable function** if some poly time TM M exists and halts with just $f(w)$ on its tape, given any input w.

i.e. M transforms w to $f(w)$ in poly time.

Definition

Language A is **poly time mapping reducible**, or simply **poly time reducible**, to language B, written $A \leq_P B$, if a poly time computable function $f : \Sigma^* \to \Sigma^*$ exists, where for every w, $w \in A \iff f(w) \in B$.
The function f is called the **polynomial time reduction** of A to B.

If a new problem A is poly time reducible to a known problem B with poly time solution, we obtain a poly time solution to A!

More precisely

Theorem

If $A \leq_P B$ *and* $B \in \mathbf{P}$, *then* $A \in \mathbf{P}$.

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$N =$ "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."

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Accounting (analysis):

- $w \in A$ whenever $f(w) \in B$ because f is a reduction from A to B
- M accepts $f(w)$ whenever $w \in A$
- N runs in poly time: each of its two stages runs in poly time.

Poly time reduction: an example

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E.g. $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2 \vee x_2)$, where $k = 3$.

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E.g. $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2 \vee x_2)$, where $k = 3$. Convert ϕ to a graph for $CLIQUE$.

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The reduction f generates the string $\langle G, k \rangle$, where G is an undirected graph defined as:

- The nodes in G are organized into k groups of three nodes each called the triples, $t_1, \ldots, t_k$.
  - Each triple corresponds to one of the clauses in ϕ, and
  - each node in a triple corresponds to a literal in the associated clause. Label each node of G with its corresponding literal in ϕ.
- The edges of G connect all but two types of pairs of nodes in G.
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We show that ϕ is satisfiable iff G has k-clique.

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Therefore G contains a k-clique.

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- Each triple contains a clique node and hence each clause contains a literal that is assigned TRUE.

Therefore ϕ is satisfiable.

Fin (wakas)

Thanks for the attention.

Questions?

Reading assignment(s)

[Sipser 2005] Chapter 7.3, 7.4 or

[Hopcroft et al 2001] Chapter 10.2

References:

[Sipser 2005] M. Sipser. *Introduction to the Theory of Computation*: 2ed. PWS Publishing Company, 2005.

[Hopcroft, Ullman 1979] J. Hopcroft, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 1979.

[Hopcroft et al 2001] J. Hopcroft, R. Motwani, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 2001.

[Hernandez 2014] CS133 lecture slides of N.H.S. Hernandez, 2014