CS 133 : Automata Theory and Computability

Lecture Slides

Time Complexity

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Day 26

NP-completeness

More on NP-completeness

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More on **NP**-completeness

Class NP

or problems solvable in polynomial time with a 1-tape NTM. More formally

Definition

Language L is in class **NP** if there is a NTM M such that $L = L(M)$ and M has polynomial run time $t(n)$.

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More real-world applications: Cryptography!

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Method to prove **NP**-completeness? Polynomial time reduction! ($\leq_p$)

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Theorem: $3SAT \leq_p CLIQUE$.

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*Once we have one **NP**-complete problem, we may obtain others by poly time reduction from it!*

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▷ Show that every language A in **NP** is poly time reducible to SAT.

Pick any language A in **NP**.
Let N be an NTM that decides A in $O(n^k)$ time for some constant k.

[1] Alternate version.

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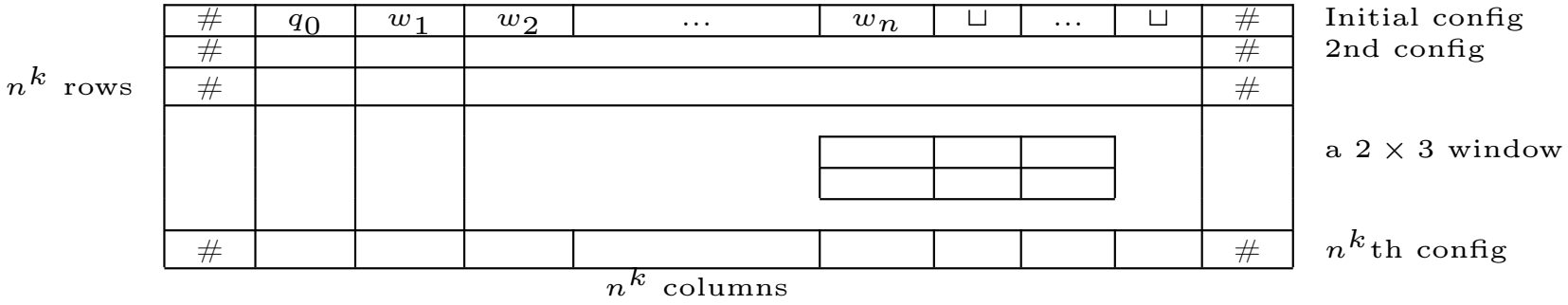

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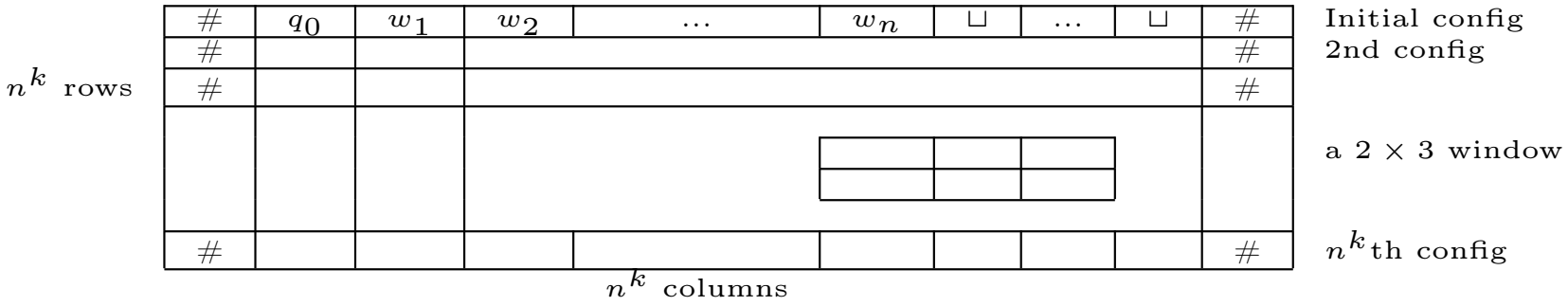

Problem of determining if N accepts $w \iff$ problem of determining if an accepting tableau for N on w exists.

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- For each $1 \leq i, j \leq n^k$ and $s \in C$, add variable $x_{i,j,s}$. $x_{i,j,s} = 1$ if $cell[i, j]$ of tableau contains s.

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- The formula $\phi_{N,w}$ is the AND of four parts, i.e.

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$$\phi_{cell} = \bigwedge_{1 \le i,j \le n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{s,t \in C, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right) \right]$$

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- 1st clause: at least one variable $x_{i,j,s} = 1$ corresponds to $cell[i,j] = s$.

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- 1st clause: at least one variable $x_{i,j,s} = 1$ corresponds to $cell[i,j] = s$.
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- Both clauses: we ensure any satisfying assignment of $\phi_{N,w}$ corresponds to one symbol in each cell of tableau.

$$\phi_{N,w} = \phi_{cell} \wedge \phi_{start} \wedge \phi_{move} \wedge \phi_{accept}.$$

Formula ϕ_{start} ensures that 1st row of tableau is the *initial configuration* of N on $w = w_1 w_2 \ldots w_n$:

$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge x_{1,n+3,\sqcup}$$

$$\wedge \cdots \wedge x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#}$$

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Formula ϕ_{accept} ensure that an *accepting configuration occurs* in tableau.

$$\phi_{accept} = \bigvee_{1 \le i,j \le n^k} x_{i,j,q_{accept}}$$

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Finally, formula ϕ_{move} ensures row i of tableau *legally follows* from row $i-1$ based on transitions of N. Use concept of *legal windows*.

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$$\phi_{move} = \bigwedge_{1 \le i,j \le n^k} (\text{ the } (i,j) \text{ window is legal})$$

Replace the text 'the (i, j) window is legal' with the following formula:

$$\bigwedge_{\substack{a_1, a_2, \ldots, a_6 \\ \text{is a legalwindow}}} \left(x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6}\right)$$

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  - Upper bound of $\phi_{N,w}$'s total size: $O(n^{2k})$.
  - This bound is good enough! Bound shows size of $\phi_{N,w}$ is polynomial in n. If more than poly, reduction f of A to SAT cannot have any chance of generating it in poly time.

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  - ϕ_{move} and ϕ_{accept} each contains a fixed-size fragment of the formula for each cell, so its size is $O(n^{2k})$.
  - Upper bound of $\phi_{N,w}$'s total size: $O(n^{2k})$.
  - This bound is good enough! Bound shows size of $\phi_{N,w}$ is polynomial in n. If more than poly, reduction f of A to SAT cannot have any chance of generating it in poly time.

- Estimate total number of variables:
  - $n^k \times n^k$ table contains n^{2k} cells.
  - Each cell has $l = |C|$ variables associated with it.
  - Since l depends only on TM N and not on input length, total number of variables is $O(n^{2k})$. Polynomial!
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Since any A in **NP** reduces to SAT, SAT is **NP**-complete.

Back to $3SAT$

Corollary

*3SAT is **NP**-complete*

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- If clause contains l variables, $(a_1 \vee a_2 \vee \cdots \vee a_l)$, we can replace it with the $l-2$ clauses
  $(a_1 \vee a_2 \vee z_1) \wedge (\overline{z_1} \vee a_3 \vee z_2) \wedge (\overline{z_2} \vee a_4 \vee z_3) \wedge \cdots \wedge (\overline{z_{l-3}} \vee a_{l-1} \vee a_l)$.
- Now we have a formula in CNF with 3 variables per clause!

NP-completeness

More on NP-completeness

NP-completeness is widespread!

NP-complete problems are found in practically any field requiring computation, e.g.

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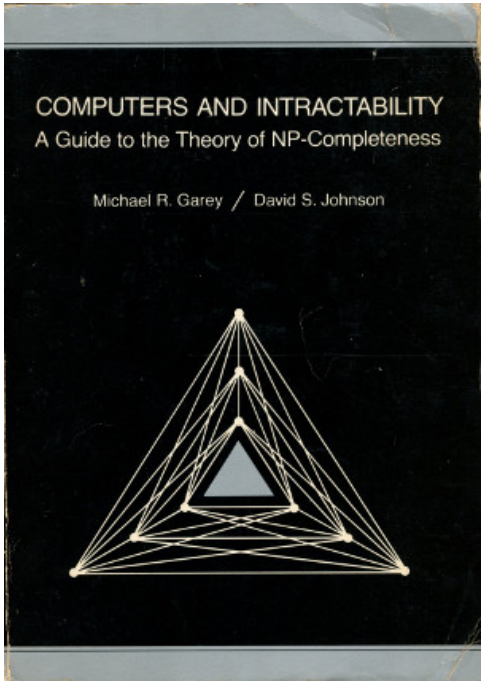

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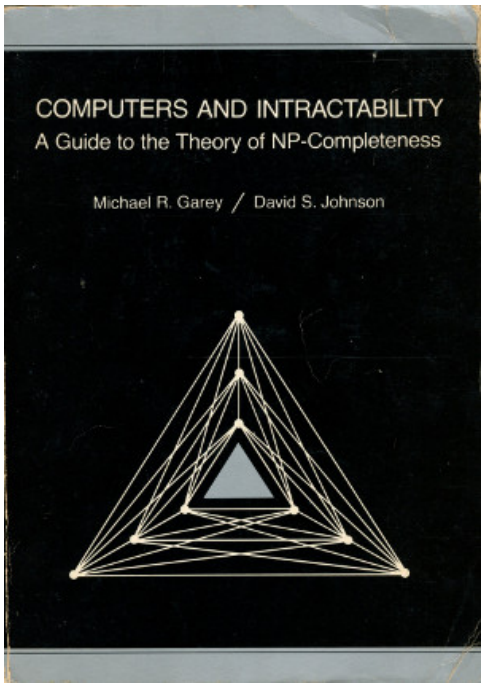

If you cannot find a poly time algo for a problem, good idea to check if problem is **NP**-complete.

Corollary

*CLIQUE is **NP**-complete.*

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The *vertex cover* (or *node cover*) problem:

A vertex cover of an undirected graph $G = (V, E)$ is a set $V' \subseteq V$ where all nodes in V' "cover" all edges of G,

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The vertex cover problem asks whether a graph contains a vertex cover of a size k:

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Need to show: $3SAT \leq_p VERTEX\text{-}COVER$.
Reduction maps a 3CNF ϕ to a graph G and number k, so ϕ is satisfiable only if G has a k-node vertex cover.

$3SAT \leq_p VERTEX\text{-}COVER$

Poly time reduction technique: from $3SAT$ *to some language* A*, look for structures in* A *that can simulate variables and clauses in Boolean formulas.*
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- Thus the total number of nodes that appear in G is $2m + 3l$ where ϕ has m variables and l clauses. Let $k = m + 2l$.

E.g. $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2 \vee x_2)$.

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- Select one TRUE literal in every clause and put the remaining two nodes from every clause gadget into the vertex cover.
- These $k = m + 2l$ nodes cover all edges. (Why?)

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- Hence, G has a vertex cover with k nodes.

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- Hence, ϕ has a satisfying assignment.

Theorem

*SUBSETSUM is **NP**-complete.*

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The reduction maps a 3CNF ϕ with variables $x_1, \cdots, x_l$ and clauses $c_1, \cdots c_k$ into an instance of the $SUBSETSUM$ problem $\langle S, t \rangle$ wherein the elements of S and the number t are the rows in the table on the next slide.

	1	2	3	4	...	l	c_1	c_2	...	c_k
y_1	1	0	0	0	...	0	1	0	...	0
z_1	1	0	0	0	...	0	0	0	...	0
y_2		1	0	0	...	0	0	1	...	0
z_2		1	0	0	...	0	1	0	...	0
$\vdots$					$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_l						1	0	0	...	0
z_l						1	0	0	...	0
g_1							1	0	...	0
h_1							1	0	...	0
g_2								1	...	0
h_2								1	...	0
$\vdots$									$\ddots$	$\vdots$
g_k										1
h_k										1
t	1	1	1	1	...	1	3	3	...	3

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The reduction maps a 3CNF ϕ with variables $x_1, \cdots, x_l$ and clauses $c_1, \cdots c_k$ into an instance of the $SUBSETSUM$ problem $\langle S, t \rangle$ wherein the elements of S and the number t are the rows in the table on the previous slide, expressed in ordinary decimal notation. Constructed table has $2l + 2k$ decimal numbers (rows) and the final row for t. Each decimal number has at most $l + k$ digits (columns).

- S contains one pair of numbers, y_i, z_i for each variable x_i in ϕ.
- The decimal representation is in two parts.
  The left-hand part comprises a 1 followed by $l - i$ 0s.
  The right-hand part contains one digit for each clause.
  - the j^{th} digit of y_i is 1 if clause c_j contains literal x_i
  - the j^{th} digit of z_i is 1 if clause c_j contains literal $\overline{x_i}$
- S also contains one pair of numbers g_j, h_j for each clause c_j.
  These numbers are equal and consist of a 1 followed by $k - j$ 0s.
- The target number t, consists of l number of 1s followed by k number of 3s.

Suppose ϕ is satisfiable.

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We construct a subset of S as follows:

- Select y_i if $x_i = 1$. Select z_i otherwise.
  - so far, the sum contains a 1 in each of the first l digits and a number between 1 and 3 in each of the last k digits
- Select enough of the g and h numbers to bring each of the last k digits up to 3.
- Sum would then be equal to t.

	1	2	3	4	...	l	c_1	c_2	...	c_k
y_1	1	0	0	0	...	0	1	0	...	0
z_1	1	0	0	0	...	0	0	0	...	0
y_2		1	0	0	...	0	0	1	...	0
z_2		1	0	0	...	0	1	0	...	0
$\vdots$					$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_l						1	0	0	...	0
z_l						1	0	0	...	0
g_1							1	0	...	0
h_1							1	0	...	0
g_2								1	...	0
h_2								1	...	0
$\vdots$									$\ddots$	$\vdots$
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h_k										1
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Note: a "carry" into next column in the table never occurs when a subset of S is added, since each column sums up to at most 5.

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We make the satisfying assignment as follows:

- To get a 1 in each of the first l columns, the subset must have either y_i or z_i for each i but not both.
  - if the subset contains y_i, assign $x_i = 1$; otherwise, $x_i = 0$.
- For each of the final k columns, at most 2 can come from g and h. To sum up to 3, thus matching the digit (column) of t, at least 1 (for each of these columns) must come from some y_i or z_i in the subset.
  - if it is y_i, then x_i appears in c_j and $x_i = 1$, so c_j is satisfied.
  - if it is z_i, then $\overline{x_i}$ appears in c_j and $x_i = 0$, so c_j is satisfied.
- Therefore, ϕ is satisfied.

More NP-complete problems

$HAMPATH, GRAPHCOLOR, KNAPSACK, LCSP$, Super Mario Bros., Candy Crush Saga, The Legend of Zelda, Pokémon et al.

`https://en.wikipedia.org/wiki/List_of_NP-complete_problems`

Fin (wakas)

Thanks for the attention.

Questions?

Reading assignment(s)

[Sipser 2005] Chapter 7.4, 7.5 or

[Hopcroft et al 2001] Chapter 10.2, 10.3, 10.4

References:

[Sipser 2005] M. Sipser. *Introduction to the Theory of Computation*: 2ed. PWS Publishing Company, 2005.

[Hopcroft, Ullman 1979] J. Hopcroft, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 1979.

[Hopcroft et al 2001] J. Hopcroft, R. Motwani, J. Ullman. *Introduction to Automata Theory, Languages, and Computation*: Addison-Wesley, 2001.

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